Monte Carlo Treatment of Data Uncertainties in Thermal Analysis

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Alternatives to "worst-case" thermal analysis are discussed, and a Monte Carlo method is developed for treating the uncertainties inherent in property data for such analysis. Comparison is made with another treatment of such problems using sensitivity analysis. Comments are given on the combined use of Monte Carlo and sensitivity analysis as an efficient and accurate way of improving thermal analysis. Comparison of various analytical treatments is given by means of an example problem.

Nomenclature

A_{ij} = area for conduction between nodes i and j
F_{ij} = configuration factor for radiation between nodes i and j
$\mathcal{F}_{ij}$ = radiation exchange factor between nodes i and j
k = thermal conductivity
R = random number in range 0–1
T = temperature
Δx_{ij} = conduction path length between nodes i and j
x_i = independent variables
y = dependent variable
ξ = variable
σ = standard deviation

Subscripts

i, j = node indices
m = mean value

Superscripts

$*$ = denotes dummy variable of integration

Introduction

THREE major sources of uncertainty can exist in a thermal analysis: uncertainty in the modeling of a system; errors in the method of solution; and uncertainty in the data used.

Uncertainty in modeling can be reduced by using the knowledge that is presently being gathered about the importance of directional and spectral effects. Semigray methods often account adequately for spectral effects in spacecraft modeling, while recent development of programs using specular-diffuse or even bidirectional surface properties allows consideration of directional property effects.

Solution methods are now developed to the point that matrix inversion routines for very large matrices and the other problem areas in finite difference techniques are fairly well under control.

One of the chief remaining error sources in thermal analysis is that of the uncertainty in the data used. These uncertainties can be in both the physical properties and in the conditions and geometry of the system being modeled, and may result from a lack of knowledge of the physics of the problem or a lack of knowledge of the required properties. For example, the emittance of an extruded aluminum surface may be available in a handbook, but the question of how well the handbook number applies to the specific surface being modeled is moot. In addition, unless a particularly simple geometry is being considered, some uncertainty exists in the radiation configuration factors, the thermal contact resistance at a joint, the effect of manufacturing tolerances on dimensions, and possibly many, many other factors. The uncertainty in the results of thermal analysis caused by such errors in input information is the concern of the remainder of this paper.

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Methods for Treating Uncertainty in Data

When uncertainty exists in the data fed into a thermal analysis, a number of ways are available to face the problem. The most common approach is the "worst-case analysis." In this approach all uncertain parameters are given extreme values that lead to sufficient overdesign to compensate for the uncertainties. If a number of parameters are uncertain, this approach can lead to gross overdesign in thermal protection systems. Resulting penalties of useless size and weight can result.

If a thermal analysis is performed on a system where experimental data has been taken, it is possible to correct the input data for the thermal model so that the predicted and experimental temperatures agree. This error correction approach for thermal networks is valuable when it can be applied, for the corrected thermal model should then be capable of accurate predictions under new conditions. Often, however, insufficient experimental test temperatures are available to make a complete correction for the entire network.

Another method of determining the effects of data uncertainties is the use of "sensitivity analysis." In this method, the statistical equation is applied for the effect of uncertainties in parameters x_i on the variable y

$$dy = (\partial y/\partial x_1)\, dx_1 + (\partial y/\partial x_2)\, dx_2 + \cdots \quad (1)$$

If $dx_1, dx_2 \ldots$ are the uncertainties in $x_1, x_2 \ldots$, then the effect of each uncertainty on y is seen by examining each term on the right. In thermal analysis, y might be the temperature at a point and the x_i are the various conduction and radiation path conductances between nodes. This technique allows the analyst to see the important network conductances that affect a given temperature and thus to see which factors introduce the most uncertainty. One objection to this method is that the analysis as previously applied to thermal modeling[1] assumes that the x_i are independent. However, if the x_i are the radiation conductances, these involve the radiation interchange factors such as the $\mathcal{F}_{ij}$. These are not independent, since a change in any $\mathcal{F}_{ij}$ introduces changes in some or all the others because of the reciprocity and summation relations between configuration factors. This dependence may cause the limited sensitivity analysis to give erroneous results, especially for small networks. Hahn and Shapiro[2] give a general treatment of sensitivity analysis, and allow inclusion of correlated input parameters.

Monte Carlo Analysis

Another way exists for carrying out property error analysis in thermal modeling, and it has none of the objections listed

for the methods already discussed. It does have other disadvantages, however. The method is to build a large number of thermal models. Each model is constructed by choosing the individual uncertain parameters from the statistical distributions of these parameters. The average outcome of the various models along with the variance or other measure of scatter in the outcome gives a statistically correct picture of the effect of the input uncertainties.

As with sensitivity analysis, the statistical (or Monte Carlo) approach ultimately requires that the analyst know the uncertainties of all input parameters. Further, if a thermal analysis must be run fifty or more times to obtain the statistical data required, then a 1500 node transient analysis carried out by this method is not within reason. Some ways to avoid these objections will now be discussed.

Determination of Data Uncertainties

The ideal way of carrying out a statistical thermal analysis would be to have the frequency distributions of all the parameters for each element of the system being modeled. This would require in-site measurements of all input parameters. The next best set of information would be to possess or assume a known form for the distribution of error and prescribe the mean values, variances (or standard deviations) and other statistical measures of all input parameters. In practice, this information is seldom if ever at hand. Occasionally, an emittance is found "$\varepsilon = 0.033 \pm 0.005$." Even this is not much help. It tells us something about the precision in measurement of the reported value, but it does not tell us the uncertainty in applying such a number to our particular model. Manufacturing tolerances, oxidation, fingerprints ... all may affect a given parameter, and the more careful we must be in specifying the mean value and its uncertainty in the Monte Carlo analysis.

Reducing the Time for Monte Carlo Analysis

Many of the input parameters used in a given thermal analysis have little effect on the results. For example, radiation between node a and node b may be unimportant because the configuration factor F_{a-b} is small, or node b has a high reflectance, or because of blockage, or for other reasons. Thus even if F_{a-b} or ρ_b or ε_a have fairly large uncertainties, the effect on the temperature of node b, T_b, may still be quite small. It follows that in building a statistical thermal model, we should pay particular attention not only to the magnitude of the uncertainties of the individual input parameters, but also to which uncertainties have the greatest effect on the results. This latter information can be obtained by sensitivity analysis. The partial derivative terms in Eq. (1) that have the largest value contain the x_i parameters with the greatest influence on y. Thus, to reduce some of the Monte Carlo run time, a sensitivity analysis should first be run on the thermal model. Even for fairly large networks, such analyses are reasonably fast running. (One example is 2 min machine time for sensitivity analysis of all thermal connections in a steady-state 50 node problem on the Univac 1108). The results allow us to disregard the influence of uncertainties in some parameters; we simply input the best value we have and assign zero variance to it. We are assured that even if this best value is wrong, the effect of any error will be small. However, the larger the influence to a given parameter, the more accurately we must define the limits of uncertainty of the value.

We are often faced with estimating the uncertainty in a value on the basis of a very small or often single sample. Statistics as a discipline is not much help in this case. Good[3] gives as good a guideline as any "... real life is both complicated and short, and we make no mockery of honest adhockery."

The "honest adhockery" can only be the engineer's best estimate of the uncertainty of a given value. Ishimoto and Bevans[1] have suggested that thermal radiation properties (emittance, reflectance, etc.) are probably known within a ± 0.03 band, thermal interface conductance in "hardware" joints are $\pm 100\%$, and configuration factors are known within a variation of $\pm 5\%$. Although there are many special cases that fall outside these limits, they are probably reasonable in the absence of better intuition. What must be realized is that the experimental error reported for a set of data is not a good indication of uncertainty for a thermal analysis. Powell,[4] for example, shows a comparison of liquid thermal conductivities reported by two or more authors as compared with the results of Challoner and Powell[5] and Challoner et al.[6] The various authors reported their individual experimental error as within a few percent. Table 1 indicates differences of up to -18 and $+50\%$ between authors. Tsederberg[7] similarly notes that reported liquid conductivities varied by $\pm 10 - 20\%$ among various authors, even among those deemed most reliable. Steigmeier[8] shows that the thermal conductivity of germanium as reported by the four authors in closest agreement varied by $\pm 15\%$, and of silicon by $\pm 12\%$ among six authors. Again, each author claimed experimental accuracy of a few percent. The experimental measurement of thermal conductivities of liquids is notoriously difficult due to nearly unavoidable free convection effects. These experimental results are presented in order to show that a single reported value and its experimental accuracy should not be accepted for use in a thermal analysis. A given material may vary from the reported value by considerably more than the reported accuracy, for whatever the reason might be. Of course, reported conductivities for the more common solid materials should be in much better agreement than for the materials previously noted.

Details of the Monte Carlo Analysis

The type of Monte Carlo analysis performed here is not new; however, application to data uncertainties in thermal analysis is. The analysis consists of randomly choosing the individual values of thermal parameters (emittance, thermal conductivity, etc.) from the distributions of parameter values. The individual values are input into the thermal model, which is then solved. This process is repeated a number of times, and the mean outcome and variance of the thermal model is found.

In general, insufficient data is available on any thermal parameter to specify the complete distribution of values. Here, it is assumed that the distribution is the normal error distribution of the form

$$f(\xi) = 1/[\sigma(2\pi)^{1/2}]e^{-(\xi-\xi_m)^2/2\sigma^2} \qquad (2)$$

where ξ_m is the mean value of the parameter ξ and σ is the standard deviation of ξ. If other forms for the distribution are found, or even if numerical data for the frequency distributions are known, these can be used in the analysis. However, most experimental data tends to have a normal error distribution, and that distribution is used here for illustration. Other distributions, including the case of correlation between the variables is treated in Ref. 9.

Table 1 Comparison of thermal conductivities of liquids measured by various investigators (from Ref. 4)

Liquid	Range of differences around values of Ref. 5.and 6
Water	-18 to $+10\%$
Glycerine	-6 to $+11\%$
Ethanol	-0.5 to $+8\%$
Toluene	$\pm 14\%$
Medicinal paraffin	$\pm 1\%$
Carbon tetrachloride	-6 to $+50\%$

To choose the value of ξ for an individual input parameter, we use the "fundamental theorem of Monte Carlo"

$$R = \int_{-\infty}^{\xi} f(\xi^*)\, d\xi^* \tag{3}$$

where R is a number chosen at random from a uniform distribution of numbers in the range 0–1. Equation (3) must be solved for ξ.

Let $x = (\xi - \xi_m)/\sigma$. Making this substitution into Eq. (2) and then substituting (2) into (3) gives a relation between R and the error function (erf) as

$$R = 1/(2\pi)^{1/2} \int_{-\infty}^{x} e^{-x^2/2}\, dx = \text{erf}\,(x) = \text{erf}[(\xi - \xi_m)/\sigma] \tag{4}$$

By knowledge of the mean value ξ_m and the standard deviation σ, the value of ξ is determined from Eq. (4) by choosing the random number R. R can be obtained from standard computer algorithms or input listings. We can rewrite Eq. (4) as

$$\xi = \xi_m \pm \sigma\, \text{erf}^{-1}\,(R) \tag{5}$$

Here, erf^{-1} is the inverse of the error function in the sense of $\sin^{-1}$; that is, $\text{erf}^{-1}\,(R)$ is the value the error function of which is R. The sign may be chosen by use of a second random number.

To illustrate the method, an example problem was chosen. The geometry is that of a proposed thermal protection system for cryogenic storage tanks on a highspeed vehicle to be used within the earth's atmosphere. The structure is composed of a 1-in. thick fiberglass honeycomb structure (Fig. 1). These layers are attached through a pad of low thermal conductivity material to an aluminum rib welded to the tank exterior. It is assumed that the rib spaces are evacuated. Finally, a 1-in. layer of foam insulation is on the tank interior, with thermal conductivity taken equal to that of the gaseous material in the tank (hydrogen). The system is assumed one-dimensional.

Radiation exchange among nodes 8–16 is possible. Configuration factors between each pair of these nodes were calculated, and the calculated values were taken as the mean values. A standard deviation was assigned to each configuration factor, using judgment and adhockery. For this particular problem, the standard deviations are taken to be small, because the factors can be calculated quite well using exact relations. It is probably justifiable to assign even larger deviations when we consider the possible presence of directional effects, the effects of manufacturing tolerances on the dimensions, fabrication errors, nonplanar (distorted) surfaces, etc.

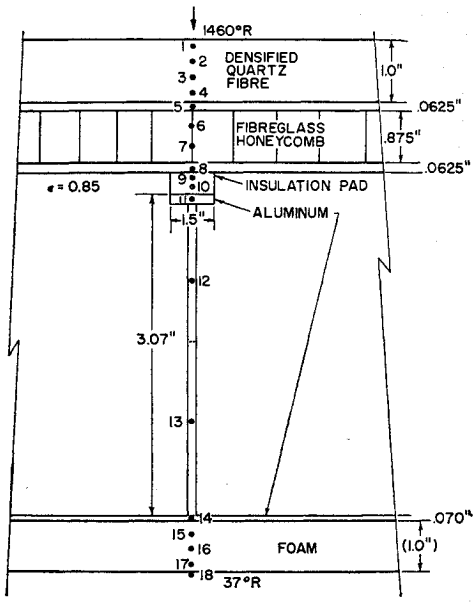

Fig. 1 Geometry of example system.

Emittances for the various nodes in the example were taken from handbook values. Again, standard deviations are taken from experience as to the range of possible values. An initial matrix inversion routine was used to calculate the radiation exchange factors (script F's or $\mathcal{F}_{ij}$). These were calculated 50 times from 50 models constructed from the input data. The resulting $\mathcal{F}_{ij}$ values were then tabulated as mean values with the standard deviation. The other required input data are given in Table 2.

An energy balance equation written for each node i assigned to the model is of the form

$$\sum_j \left[\frac{k_{ij}A_{ij}}{\Delta x_{ij}}(T_i - T_j) + \mathcal{F}_{ij}(T_i^4 - T_j^4)\right] = 0 \tag{6}$$

where k_{ij} is the thermal conductivity for the material between nodes i and j, A_{ij} is the area of the conduction path and Δx_{ij} is the length of the path from i to j, and $\mathcal{F}_{ij}$ is a radiative

Table 2 Input data for example problem

Nodes	Thermal conductivities		Conductance areas		Conductance paths	
ij	k_{ij}, Btu/hr ft °R	$\sigma(k_{ij})$	A_{ij}, ft²	$\sigma(A_{ij})$	Δx_{ij}, ft	$\sigma(\Delta x_{ij})$
0–1	0.042	0.005	1.66	0.01	0.0105	0.0001
1–2	0.041	0.005	1.66	0.01	0.0208	0.0002
2–3	0.040	0.005	1.66	0.01	0.0208	0.0002
3–4	0.039	0.005	1.66	0.01	0.0208	0.0002
4–5	0.037	0.005	1.66	0.01	0.0130	0.0003
5–6	0.22	0.05	0.023	0.002	0.0269	0.0006
6–7	0.22	0.05	0.023	0.002	0.0243	0.0005
7–8	0.22	0.05	0.023	0.002	0.0269	0.0006
8–9	0.075	0.02	0.064	0.005	0.0109	0.0002
9–10	0.030	0.008	0.125	0.005	0.025	0.002
10–11	18.0	2.0	0.125	0.005	0.011	0.001
11–12	70.0	5.0	0.0089	0.0002	0.065	0.003
12–13	60.0	5.0	0.00583	0.0002	0.125	0.005
13–14	50.0	5.0	0.0629	0.0005	0.065	0.003
14–15	6.1	0.5	1.66	0.01	0.024	0.001
15–16	0.04	0.01	1.66	0.01	0.021	0.001
16–17	0.03	0.01	1.66	0.01	0.021	0.001
17–18	0.015	0.005	1.66	0.01	0.021	0.001

$T_0 = 1460$ °R, $T_{18} = 37$ °R — Temperatures of boundary nodes

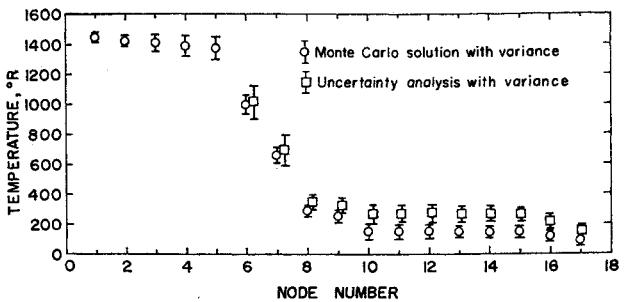

Fig. 2 Temperature distribution for example problem.

exchange factor between nodes i and j. $\mathcal{F}_{ij}$ depends upon the configuration factor F_{ij} between i and j, the emissivities of i and j, and the area of i.

The matrix of j equations is solved for T_j. This is repeated 50 times with sets of properties chosen by inserting the mean value and standard deviation for each property into Eq. (5), and then inserting a random number and calculating the property.

Figure 2 shows the temperature distributions computed by the Monte Carlo approach, and comparison is made with the sensitivity analysis results from the method of Ref. 1. The vertical lines show the predicted standard deviation of the results by virtue of the uncertainties in input data. It should be noted that the methods are in excellent agreement until nodes are reached where the radiative transfer effects are significant. This disagreement may be due to the assumption in the sensitivity analysis that the $\mathcal{F}_{ij}$ values are independent.

In Fig. 3, a comparison is given of the temperature differences computed by different methods of analysis. The statistical Monte Carlo method, using the variances of the thermal properties and assuming a normal distribution, is used as a base. The solid line shows the difference between the Monte Carlo mean temperature values, and the temperatures calculated by a worst case analysis. The worst case analysis was carried out using the mean property values of Table 2, and adding or subracting twice the standard deviation shown in Table 2 so as to drive the heat transfer in the system toward the maximum values. This gave temperatures for the worst case that were always smaller than the Monte Carlo results, the maximum difference being 35° at node five.

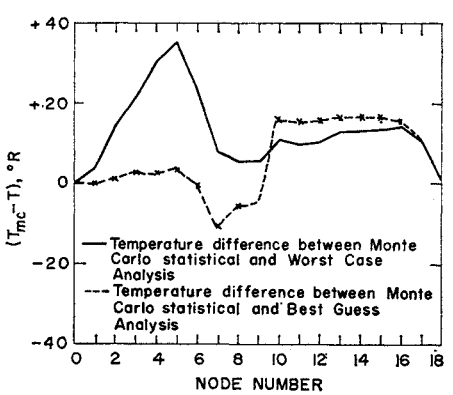

Fig. 3 Comparison of temperatures calculated by various methods of thermal analysis.

The dashed line on Fig. 3 shows the difference between the temperatures calculated by the Monte Carlo statistical technique and those calculated using a "best guess" for the thermal properties. The best guess is taken to be the mean values in Table 2. The differences are within $\pm 17°$.

Both the worst case and the best guess analyses show an increased temperature gradient for nodes 15–18. Thus the statistical analysis for this particular analysis predicts a lower heat rate into the last node of the thermal model than a worst case analysis or the analysis using mean property values.

Summary

Statistical techniques are developed to show how a more accurate thermal analysis can in principle be carried out. Difficulties in time, effort and accurate data for input to the method are discussed. Results of two such techniques are compared, and shown as alternatives to worst-case analysis.

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